Synchronous versus asynchronous updating in the ''game of Life''

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The rules for the ''game of Life'' are modified to allow for only a random fraction of sites to be updated in each time step. Under variation of this fraction from the parallel updating limit down to the Poisson limit, a critical phase transition is observed that explains why the game of Life appears to obey self-organized criticality. The critical exponents are calculated and the static exponents appear to belong to the directed percolation universality class in 2+1 dimensions. The dynamic exponents, however, are nonuniversal, as seen in other systems with multiple absorbing states. [S1063-651X(99)13903-5]

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In mathematical modeling of complex, many-particle systems the issue of updating arises. Typically the model seeks to emulate the temporal evolution of some natural process, either analytically or via simulation. In nature time is a continuous variable but in mathematical models it is often more convenient to discretize time. Computer simulations, in particular, must discretize time because all numbers are represented by a fixed number of binary digits.

How time is discretized tends to depend on the system being modeled and the preferences of the modeler. Different fields have adopted one of two techniques. Ecologists, for example, observe the natural periodicity in the population(s) being studied and choose time steps on the same scale [1]. Such a model for the number of individuals N_i of species i might look like

$$N_{i,t+1} = \mu_i N_{i,t} \left(1 - \frac{N_{i,t}}{K_i} \right) + \sum_j M(N_{i,t}, N_{j,t}),$$

where the indices i and j range over the number of species being modeled, the first term represents some internal single species dynamics for estimating the change in the population over one cycle, and the second term represents the effects of species interactions. Time is represented by the integer t that counts the number of cycles (such as years) that have passed. This type of updating is called *parallel* or *synchronous* updating. It is also commonly found in coupled map lattices meant to describe intermittency in turbulent flow [2].

In nature the probability that two events occur at exactly the same time has measure zero because time is infinitely divisible. Some models take this into account by updating each individual component of the model separately. Update times for all the elements are calculated and the element with the shortest time is updated first, and then the update times are recalculated as needed and the process repeated. This type of updating is called *asynchronous* updating. If all the elements in a model can be updated at random times, but each has a known average rate then they are Poisson processes and the updating scheme is often called Poisson updating. Other types of asynchronous updating methods include sequential updating and combinations where some elements are updated in a fixed sequence and others randomly.

The question naturally arises: Does it matter which type of updating is chosen? There is significant evidence that it does matter indeed. Many different finite-differencing schemes have been developed for numerical solutions to initial value partial differential equations (PDE's) in order to try and avoid instabilities and spurious artifacts, introduced by discretization, not found in the original PDE's [3]. Further, maps (diffeomorphisms) often exhibit more complexity than their equivalent PDE. Consider the logistic map $x_{i+1} = \mu x_t (1 - x_t)$, which exhibits a ''period-doubling route to chaos'' as μ increases from 3 to 4. The corresponding logistic differential equation $\partial_t x = \mu x (1-x) - x$, on the other hand, has only a single stable equilibrium [$x=0$ for $\mu<1$, and $x=(\mu-1)/\mu$ for $\mu>1$]. Indeed, the differential equation does not exhibit the same complexity seen in the map.

Other illustrations of the significance of updating schemes can be found in Refs. [4–7].

In this paper we explore the effects of synchronous and asynchronous updating in one particular model based on the ''game of Life,'' a cellular automaton (CA) invented by John Horton Conway and made famous by Martin Gardner in the 1970s (Gardner's articles have been collected in [8]).

The game of Life (GL) is defined on a square two-dimensional lattice of sites which are either *alive* or *dead*. The lattice is (traditionally) updated synchronously and the rules governing the evolution vaguely mimic logistic dynamics: (1) a live site will remain alive if exactly two or three of its eight nearest neighbors are alive, otherwise it will die. (2) If a dead site has exactly three live neighbors, it will be toggled to the live state (birth). A random initial population will evolve in a complex manner over time and will eventually settle down to a steady-state configuration which is very sensitive to small perturbations.

The game of Life has enjoyed renewed interest since it was proposed that it exhibits self-organized criticality (SOC) [9,10], a description of systems that naturally tend toward a critical state (lacking any natural length scales) without requiring any tuning of external parameters. Recent evidence suggests GL is actually slightly subcritical [11–13].

In this paper we consider the effects of relaxing the synchronous updating requirement in GL. Instead of updating all

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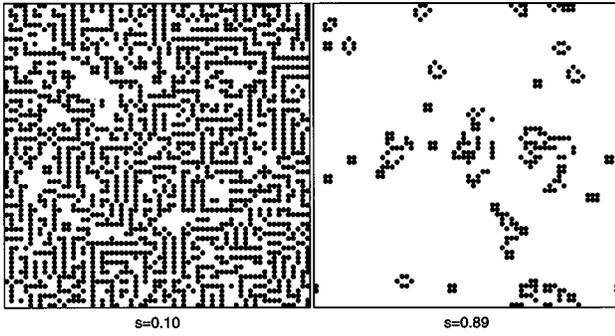†Electronic address: birger@physics.ubc.ca

FIG. 1. Sample steady-state configurations for two simulations on a 64×64 lattice with periodic boundaries. At $s=0.10$ (left) a dense state of live sites (●) organizing into regions of vertical and horizontal stripes can be discerned. In contrast, $s=0.89$ (right), just below the critical point, shows a largely stationary backdrop of relatively low density, with isolated *avalanches* of high activity.

the sites in parallel, each site is updated with some probability s. We label our new model SGL(s). If $s=1$ we recover the traditional GL [SGL(1)≡GL], but as $s \to +0$ the updating becomes Poisson. To keep the average rate of events uniform time is rescaled by s. All simulations were run on square lattices with periodic boundaries from initially random, 50% occupied configurations. (Evidence indicates that periodic boundary conditions tend to minimize finite-size effects [11,13].) The final equilibrium state is very robust with respect to the initial density; simulations starting with 25%–99% initial occupation yielded indistinguishable final states. The time to equilibrate, however, does appear to exhibit a weak dependence on the initial density.

Two distinct phases are observed as s is varied: for large s the system eventually decays to a low-density frozen state. The one exception is $s=1$ which also allows simple periodic structures to survive, but is identical in all other respects. On the other hand, for small s the system converges (after an initial transient) to a randomly fluctuating steady state. (Of course, for finite simulations the system must always drop into the absorbed state eventually, but in simulation the steady state was stable for periods much longer than the transient.) As $s \to +0$ the steady-state is characterized by domains of alternating *dead* and *live* stripes. Most of the activity in this case occurs at the domain boundaries. Typical snapshots of the steady state for low and moderately high s values are given in Fig. 1. For some intermediate s there is a transition from one phase to the other.

We choose two typical order parameters to visualize the phase transition: the density of live sites and the activity (fraction of updated sites which flip states), both of which exhibit distinct time evolutions for high and low s values. The absorbed state (for large s) has no active sites $\alpha=0$ and a low density $\rho_0=0.026\pm0.001$ [slightly lower than SGL(1) because of the lack of oscillators] while the fluctuating state is characterized by a very active, high-density population.

As s is varied the density ρ of live sites (neglecting the transient) exhibits a distinct second-order (critical) phase transition as demonstrated in Fig. 2. The critical point s_c and critical power-law exponent β can be estimated via a nonlinear Levenberg-Marquardt fit [3] of a power law $\rho-\rho_0 \propto (s_c-s)^\beta$ where $\rho_0=0.026$ as calculated above. Notice that in the Poisson limit ($s \to 0$) the steady-state density does not
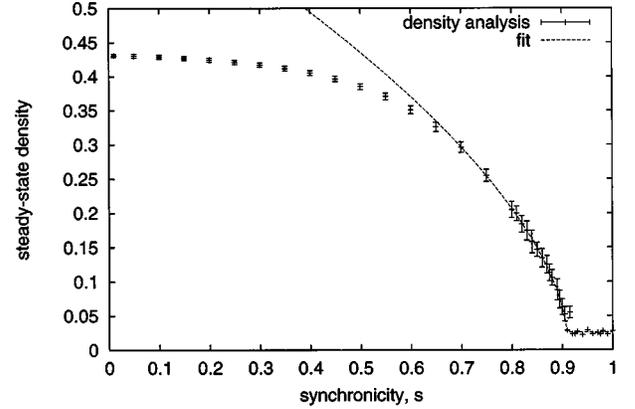

FIG. 2. The steady-state density of live sites clearly shows a critical phase transitions as the fraction of sites updated s is varied. A power-law fit (dotted curve) near the transition reveals a critical point $s_c=0.9083\pm0.0010$ and a critical exponent $\beta=0.617\pm0.011$ for simulations on a 128×128 lattice with periodic boundaries.

approach the mean-field prediction of $\rho=0.37$ [14].

Similarly, the steady-state activity α undergoes a critical transition as shown in Fig. 3 from which the critical point and exponent β' can again be estimated via a nonlinear power-law fit $\alpha \propto (s_c-s)^{\beta'}$.

To eliminate finite-size effects a scaling analysis is required. Data were collected from nine runs on square lattices with sides ranging from $L=32$ to $L=512$. The critical point, density exponent, and activity exponent were computed as above for these lattices and are plotted in Fig. 4, 5, and 6. These values appear to converge for the larger lattices suggesting boundary effects are minimized. It appears that in the thermodynamic limit the values converge to $s_c^{(\infty)}=0.9060\pm0.0004$, $\beta^{(\infty)}=0.595\pm0.004$ (density exponent), and $\beta'^{(\infty)}=0.595\pm0.006$ (activity exponent). The error margins may have been underestimated because only nine different lattice sizes were used.

The compatibility of these exponents with directed perco-
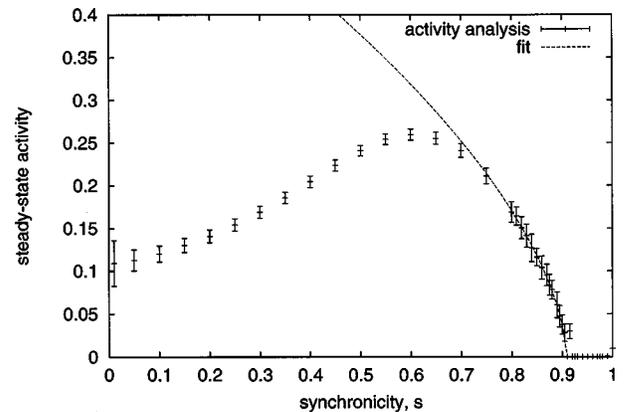

FIG. 3. The steady-state activity, the fraction of updated sites that flip states in each time step, also shows a critical phase transition as the fraction of sites updated s is varied. In this case the fitted power law (dotted curve) predicts $s_c=0.9085\pm0.0015$ and $\beta'=0.599\pm0.026$ for simulations on a 128×128 lattice with periodic boundaries. No activity is possible in the absorbed state (to the right of the critical point) except when $s=1$ (GL).
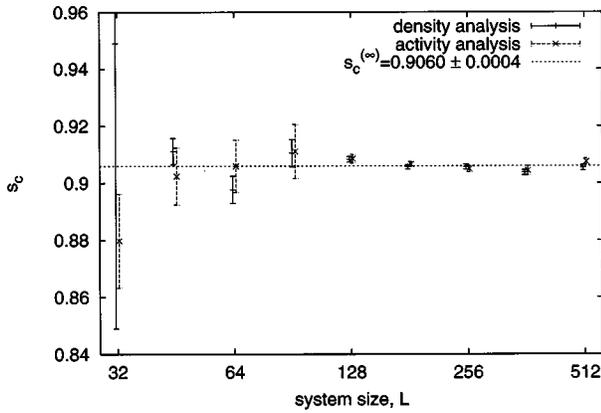
FIG. 4. As the lattice dimension L is scaled the critical point, computed from both the density and activity analyses, converges (dotted line) to $s_c^{(\infty)}=0.9060\pm0.0004$ for simulations on square lattices with periodic boundaries. The data points have been shifted slightly in the plot to improve readability.

FIG. 6. As the lattice dimension L is scaled the activity exponent converges (dotted line) to $\beta'^{(\infty)}=0.595\pm0.006$ for simulations on square lattices with periodic boundaries.

lation (DP) in $2+1$ dimensions, $\beta=\beta'=0.586\pm0.014$ [15,16], leads us to hypothesize that SGL(s_c) belongs to the DP (Reggeon field theory) universality class. This suggests that many of the details, including the particular game of Life rules, are irrelevant near the critical point.

Up to this point we have only discussed the steady-state behavior, neglecting the initial transient. But near a critical point the transient should also reveal dynamical critical behavior. The probability that an avalanche survives for time t near a critical point is $P_t=t^{-\delta}\phi((s-s_c)t^{1/\nu_\parallel})$, where δ is the survival exponent, $\nu_\parallel$ is the temporal correlation coefficient, and ϕ is an unknown scaling function [17].

To recover the dynamical exponents we require the plots of $P_t t^\delta$ vs $(s-s_c)t^{1/\nu_\parallel}$ for different values of s (but a single L value) to collapse onto one another. A sample of the collapsed data for $L=91$ is shown in Fig. 7.

The results of 400 experiments involving $s=0.92$ and $s=0.93$ on lattice sizes of $L=64$, 91, and 128 using the previous calculated $s_c^{(\infty)}=0.906$ indicate exponents $\delta=0.25\pm0.04$ and $\nu_\parallel=0.93\pm0.02$. In contrast, the directed percolation exponents in $2+1$ dimensions are $\delta=0.451\pm0.003$ and $\nu_\parallel=1.295\pm0.006$ [17]. It is not surprising that these values do not agree; evidence of such nonuniversality in the dy-

namical exponents has been observed in other critical behavior. The key ingredient for this to occur appears to be the possibility of the system settling into one of many possible absorbing states [16].

In summary, we modified the rules for the game of Life (GL) to allow for only a random fraction s of sites to be updated in each time step. As s was decreased from GL ($s=1$) a critical transition was observed that explains why GL appears to obey self-organized criticality (SOC)—in fact it is just close to a traditional dynamical critical point at $s_c\approx0.906$.

The distinct behavior of high- and low-s simulations underscores the importance of choosing a relevant updating scheme. Often, either parallel or Poisson updating is chosen for convenience, but to avoid spurious artifacts the modeler should explore alternate schemes where possible.

Some SOC models can be directly mapped onto ordinary critical models with an explicit control parameter. In some cases, such as the sand-pile model, the control parameter is the driving rate, which is reduced to zero [18]. In other cases variation of the conservation law moves the system between subcritical and supercritical regimes [19]. Our results indi-
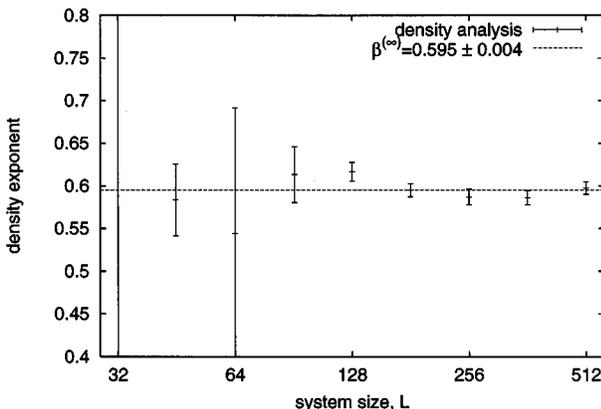

FIG. 5. As the lattice dimension L is scaled the density exponent converges (dotted line) to $\beta^{(\infty)}=0.595\pm0.004$ for simulations on square lattices with periodic boundaries.
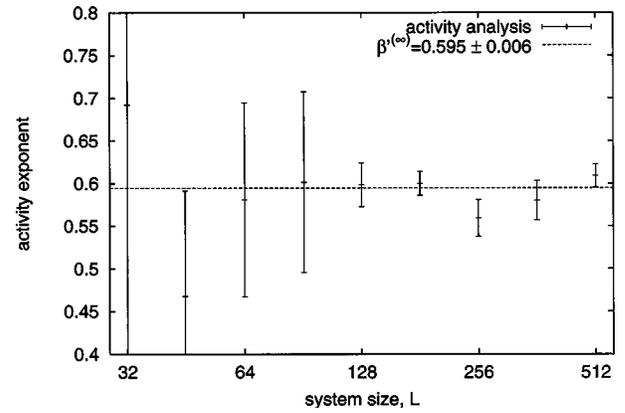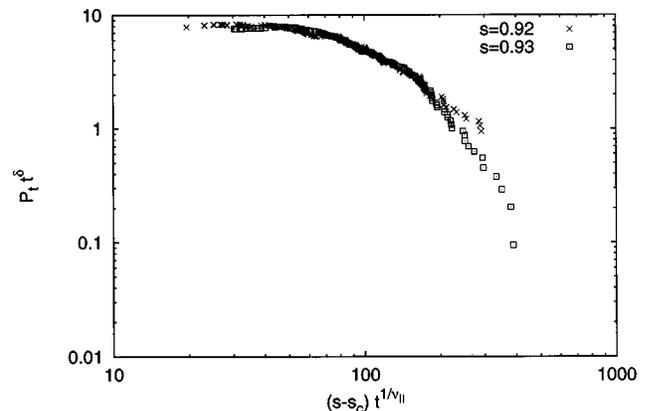

FIG. 7. By tuning the exponents δ and $\nu_\parallel$ the plots of $P_t t^\delta$ vs $(s-s_c)t^{1/\nu_\parallel}$ for different values of s can be overlapped, yielding the critical exponents (where P_t is the probability of a disturbance surviving for t iterations). This plot represents the collapsed plots for $L=91$ using $s_c=0.906$. The tails of the distributions ($P_t t^\delta$ small) are subject to excessive noise and are neglected in the fit.

cate that GL falls into this subset of SOC: GL's inclusion in the SOC paradigm results from an accidental choice of the control parameter (synchronicity) at a value near the critical point.

The compatibility of the critical exponents suggests that GL belongs to the directed percolation universality class in $2+1$ dimensions as was first suggested by Paczuski, Maslov, and Bak [20]. Nordfalk and Alström [12] also observed compatibility for different (temporal) exponents. This behavior should not be too surprising because the spread of a perturbation in a nearly stable configuration looks qualitatively very much like a directed percolation cluster.

Our own exploration of the dynamical behavior uncovered some surprising results: namely, that the dynamic critical exponents do not fall into the directed percolation universality class. Our model exhibits a characteristic property of nonuniversal critical behavior, the existence of multiple absorbing states. Further research in this area is required to explain both the nature of the phase transition and the interesting behavior of GL under Poisson updating.

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